ELSEVIER

Discrete Mathematics 258 (2002) 361–371

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Total domination supercritical graphs with respect to relative complements

Teresa W. Haynes[a,*], Michael A. Henning[b,1], Lucas C. van der Merwe[c]

[a] Department of Mathematics, East Tennessee State University, Box 70663, Johnson City, TN 37614, USA

[b] Department of Mathematics, University of Natal, Private Bag X01, Pietermaritzburg 3209, South Africa

[c] Division of Mathematics and Science, Northeast State Community College, Blountville, TN 37617, USA

Received 18 April 2000; received in revised form 19 April 2002; accepted 22 April 2002

Abstract

A set S of vertices of a graph G is a total dominating set if every vertex of $V(G)$ is adjacent to some vertex in S. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. Let G be a connected spanning subgraph of $K_{s,s}$, and let H be the complement of G relative to $K_{s,s}$; that is, $K_{s,s} = G \oplus H$ is a factorization of $K_{s,s}$. The graph G is k-supercritical relative to $K_{s,s}$ if $\gamma_t(G) = k$ and $\gamma_t(G + e) = k - 2$ for all $e \in E(H)$. Properties of k-supercritical graphs are presented, and k-supercritical graphs are characterized for small k.

Keywords: Domination; Total domination; Total domination edge critical graphs; Total domination supercritical graphs; Relative complement

1. Introduction

For terminology not defined here, we refer the reader to [6]. In particular, for a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S and is a *total dominating set* if every vertex in V is adjacent to a

* Corresponding author.

E-mail address: haynes@etsu.edu (T.W. Haynes).

[1] Research supported in part by the South African National Research Foundation and the University of Natal.

PII: S0012-365X(02)00537-X

vertex in S. The domination number and the total domination number of a graph G are denoted by $\gamma(G)$ and $\gamma_t(G)$, respectively. If S is a minimum dominating (minimum total dominating) set, we call S a *γ-set* (*γ_t-set*) of G. For sets $S, T \subseteq V$, we say S *dominates* (respectively, *totally dominates*) T if every vertex in T (respectively, $S \cup T$) has a neighbor in S, and we write $S \succ T$ (respectively, $S \succ_t T$) if S dominates T (respectively, S totally dominates T). If $T = V - S$, then we say $S \succ G$ (respectively, $S \succ_t G$). If $S = \{s\}$, we write $s \succ T$ and $s \succ G$. For a more detailed treatment of domination-related parameters and for terminology not defined here, the reader is referred to [2,6].

A graph G is said to be *γ-domination critical*, or just *γ-critical*, if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for every edge e in the complement $\bar{G}$ of G. This concept of γ-critical graphs has been studied by, among others, Blitch [1], Summer [10], Sumner and Blitch [11], and Wojcicka [13]. Haynes et al. [8,9] introduced and studied the *total domination edge critical graphs*, that is, graphs G such that $\gamma_t(G+e) < \gamma_t(G)$ for any edge $e \in E(\bar{G})$. The addition of an edge can change the domination number by at most one. However, the addition of an edge can change the total domination number by as much as two.

Proposition 1 (Haynes [8]). *If G is a graph with no isolated vertex, then for any edge $e \in E(\bar{G})$,*

$$\gamma_t(G) - 2 \leqslant \gamma_t(G+e) \leqslant \gamma_t(G).$$

The graphs G with the property $\gamma_t(G+e) = \gamma_t(G) - 2$ for any edge $e \in E(\bar{G})$ are called *supercritical*. It is shown in [9] that a graph G is supercritical if and only if G is the union of two or more nontrivial complete graphs.

If G is a spanning subgraph of F, then the graph $F - E(G)$ is the *complement of G relative to F* with respect to a fixed embedding of G into F. The idea of a relative complement of a graph was suggested by Cockayne [3] and is studied in [5]. We shall assume that the complete bipartite graph $K_{s,s}$ has partite sets $\mathscr{L}$ and $\mathscr{R}$ (representing "left" and "right"), and that $G \oplus H = K_{s,s}$ is a factorization of $K_{s,s}$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) We denote the relative complement H of G by $\bar{G}$. (The rest of this paper deals only with relative complements, so confusion with complements in the ordinary sense is unlikely.) Throughout this paper, G will be a connected spanning subgraph of $K_{s,s}$, and so H is unique.

Haynes and Henning [7] studied domination critical graphs with respect to the relative complement, that is, the graphs G such that $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for all $e \in E(\bar{G})$. In this paper, we study the same concept for total domination. We say that a graph G is *total domination edge critical relative to $K_{s,s}$* if $\gamma_t(G+e) < \gamma_t(G)$ for any edge $e \in E(\bar{G})$. Obviously, since Proposition 1 considers adding an arbitrary edge from the ordinary complement, it also applies to adding an edge from the relative complement. We note that adding an edge to a bipartite graph G from its relative complement can change the total domination number by 0,1, or 2. For example, the path P_6: $u_1, u_2, u_3, u_4, u_5, u_6$ is a subgraph of $K_{3,3}$ where all three possibilities occur. In particular, $\gamma_t(P_6 + u_1u_6) = \gamma_t(P_6) = 4$, $\gamma_t(P_6 + u_3u_6) = \gamma_t(P_6) - 1 = 3$, and $\gamma_t(P_6 + u_2u_5) = \gamma_t(P_6) - 2 = 2$.

If G is a connected spanning subgraph of $K_{s,s}$, and $\gamma_t(G)=k$ and $\gamma_t(G+e)=k-2$ for all $e\in E(\bar{G})$, then we say that G is *k-supercritical relative to $K_{s,s}$*. Although the supercritical graphs relative to ordinary complements are disconnected graphs and were straightforward to characterize in [9], there exist k-supercritical graphs relative to $K_{s,s}$ and obtaining a characterization for them appears to be difficult. Hence the motivation for this paper. We consider k-supercritical graphs relative to $K_{s,s}$ for small values of k. Since for any graph G with no isolated vertices, $\gamma_t(G)\geqslant 2$, it follows that there are no k-supercritical graphs for $k=2$ or $k=3$. Hence in what follows, we assume $k\geqslant 4$. Also, since the rest of this paper deals only with relative complements, we will omit the phrase "relative to $K_{s,s}$" unless a specific value of s needs to be mentioned.

Properties of k-supercritical graphs are presented in Section 2. In Section 3 we present k-supercritical graphs having large and small diameters. In particular, for k even, an infinite class of k-supercritical graphs of diameter $k-1$ and an infinite class of these graphs of diameter 5 are given. In Section 4, we investigate k-supercritical graphs for small k. A list of questions which we have yet to settle is given in Section 5.

2. Preliminary results

In this section, we present five lemmas that will be useful in what follows.

Lemma 2. *If G is a k-supercritical graph, then for each $uv\in E(\bar{G})$, every γ_t-set of $G+uv$ contains both u and v. Furthermore, in any γ_t-set S of $G+uv$, u (respectively, v) is the only vertex in S adjacent to v (respectively, u).*

Proof. Let G be a k-supercritical graph, and let S be a γ_t-set of $G+uv$. Then $|S|=k-2$. Since S is not a total dominating set of G, at least one of u and v belongs to S. Suppose $u\in S$ but $v\notin S$. Then u is the only vertex of S that is adjacent to v, for otherwise S would be a total dominating set of G. Let v' be any neighbour of v in G. Since v' is adjacent to at least one vertex of S, $S\cup\{v'\}$ is a total dominating set of G of cardinality $k-1$, which contradicts the fact that $\gamma_t(G)=k$. Hence, if $u\in S$, then $v\in S$. Similarly, if $v\in S$, then $u\in S$. Thus, S contains both u and v.

Suppose u is adjacent to some vertex of S different from v. Let v' be any neighbor of v in G. Then $S\cup\{v'\}$ is a total dominating of G of cardinality at most $k-1$, which contradicts the fact that $\gamma_t(G)=k$. Hence, v is the only vertex of S that is adjacent to u. Similarly, u is the only vertex of S that is adjacent to v. □

Lemma 3. *If G is a k-supercritical graph, then $\Delta(G)\leqslant s-1$.*

Proof. Suppose $x\in\mathscr{L}$ has degree s in G. Since $\gamma_t(G)=k\geqslant 4$, each vertex in $\mathscr{R}$ has degree at most $s-1$. Hence, by the Pigeonhole principle, there is a vertex of $\mathscr{L}$, say u, that is not adjacent to two vertices of $\mathscr{R}$, say v and w. We now consider the graph $G+uv$. Let S be a γ_t-set of $G+uv$. Then, $|S|=k-2$ and, by Lemma 2, $u,v\in S$. Since some element of $\mathscr{L}$ must be in S to dominate w, we may assume that $x\in S$ as otherwise we could replace any such element of S by x. Let u' be any neighbor of u

in G. Then $(S-\{u\})\cup\{u'\}$ is a total dominating set of G of cardinality $k-2$, which contradicts the fact that $\gamma_t(G)=k$. Hence, $\Delta(G)\leqslant s-1$. □

Lemma 4. *If G is a k-supercritical graph, then for any vertex $v\in V(G)$, there exists a set of $k-2$ vertices that totally dominates $G-N[v]$.*

Proof. Let G be a k-supercritical graph. Lemma 3 implies that $uv\in E(\bar{G})$ for some vertex u. By Lemma 2, there exists a set D of cardinality $k-4$ such that $D\cup\{u,v\}$ is a total dominating set of $G+uv$. Furthermore, no vertex of D is in $N(u)\cup N(v)$. Thus, $D\cup\{u,x\}$ where $x\in N(u)$ is a total dominating set for $G-N[v]$. □

Lemma 5. *For any pair of vertices u and v in a k-supercritical graph, if $N(u)\cap N(v)\neq\emptyset$*, then $N(u)\neq N(v)$.

Proof. Let u and v be vertices in a k-supercritical graph G, and assume that $N(u)=N(v)\neq\emptyset$. Necessarily, u and v are in the same partite set, say $\mathscr{L}$. By Lemma 3 we know that $\Delta(G)\leqslant s-1$, so there exists a vertex $x\in\mathscr{R}$ that is not adjacent to u or v. Then Lemma 2 implies that any γ_t-set S of $G+ux$ contains both u and x and no vertex in $N(u)\cup N(x)$. Thus, no neighbor of v is in S, so v is not totally dominated by S, a contradiction. □

Lemma 6. *If G is a k-supercritical graph, then* $\delta(G)\geqslant 2$.

Proof. Let G be a k-supercritical graph, and suppose to the contrary that G has an endvertex u. Since G is connected, Lemma 5 implies that there exists a vertex, say x, at a distance 3 from u. Let u,v,w,x be a path from u to x. By Lemma 2, any γ_t-set S of $G+ux$ contains both u and x and neither v nor w. But then $(S-\{u\})\cup\{v,w\}$ is a total dominating set of G with cardinality $k-1$, contradicting the fact that $\gamma_t(G)=k$. □

Lemma 7. *If G is a k-supercritical graph, then* $\mathrm{diam}(G)\geqslant 3$.

Proof. If $\mathrm{diam}(G)=2$, then $G\cong K_{s,s}$, and so $\gamma_t(G)=2$, a contradiction. Hence, $\mathrm{diam}(G)\geqslant 3$. □

3. Supercritical graphs with large and small diameters

In this section, our aim is to show that for each $k\geqslant 2$ and each $s\geqslant 2k-1$, there exists an infinite class of $2k$-supercritical graphs of diameter $2k-1$. On the other hand, we also show that for each $k\geqslant 2$, there exists an infinite class of $(2k+2)$-supercritical graphs of diameter 5. An interesting consequence of this section proof that j-critical graphs exist for all even values of j.

First, we construct an infinite class $\mathscr{G}$ of $2k$-supercritical graphs such that $\gamma_t(G)=2k$ and $\mathrm{diam}(G)=2k-1$ for each $G\in\mathscr{G}$. For $k\geqslant 2$ consider two copies of the path P_{2k}

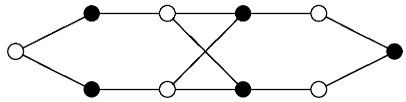

Fig. 1. A 6-supercritical graph relative to $K_{5,5}$ with diameter 5.

with respective vertex sequences $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ and $c_1, d_1, c_2, d_2, \ldots, c_k, d_k$. Let $\mathscr{K} = \{1, 2, \ldots, k\}$. Identify a_1 and c_1 and identify b_k and d_k. For each $i \in \mathscr{K} - \{1, k\}$, join a_i to d_i and b_i to c_i. Let G_k denote the resulting graph and let $\mathscr{G}$ be the family of all such graphs G_k. Note that $G_2 \cong C_6$, while the graph G_3 is illustrated in Fig. 1.

Clearly, G_k is a bipartite graph with $\mathrm{diam}(G_k) = 2k - 1$ and with partite sets $\{a_1, a_2, \ldots, a_k, c_2, \ldots, c_k\}$ and $\{b_1, b_2, \ldots, b_k, d_1 \ldots, d_{k-1}\}$ each of cardinality $2k - 1$. Hence, G_k is a spanning subgraph of $K_{s,s}$ where $s = 2k - 1$. We show that G_k is a $2k$-supercritical graph relative to $K_{2k-1,2k-1}$.

Theorem 8. *For each $k \geqslant 2$, G_k is a $2k$-supercritical graph relative to $K_{2k-1,2k-1}$ of diameter $2k - 1$.*

Proof. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $V_1 = \{a_1, b_1, d_1\}$, $V_k = \{a_k, b_k, c_k\}$, and $V_i = \{a_i, b_i, c_i, d_i\}$ for each $i \in \mathscr{K} - \{1, k\}$.

Claim 1. *For each $k \geqslant 2$, $\gamma_t(G_k) = 2k$.*

Proof. The set $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ is a total dominating set of G_k, and so $\gamma_t(G_k) \leqslant 2k$. Suppose $\gamma_t(G_k) < 2k$. Then for each γ_t-set X of G_k, $|V_i \cap X| \leqslant 1$ for at least one $i \in \mathscr{K}$. Let $I_X = \{i \in \mathscr{K}\colon\ |V_i \cap X| \leqslant 1\}$ and assume that among all γ_t-sets of G_k, X has been chosen such that $|I_X|$ is minimum.

If $V_1 \cap X = \emptyset$, then a_1 is not dominated. If $|V_1 \cap X| = 1$, then either b_1 or d_1 is in X, for otherwise X contains an isolated vertex. Say $b_1 \in X$. To totally dominate b_1, we have $a_2 \in X$; to dominate d_1, we have $c_2 \in X$ and to totally dominate c_2, $X \cap \{b_2, d_2\} \neq \emptyset$. But then $Y = (X - \{a_2\}) \cup \{a_1\}$ is a γ_t-set of G_k with $|I_Y| < |I_X|$, a contradiction. Thus, we may assume that by symmetry, $I_X \cap \{1, k\} = \emptyset$.

Hence, assume $V_i \cap X = \emptyset$ for some $i \in \mathscr{K} - \{1, k\}$. To dominate V_i, $\{b_{i-1}, d_{i-1}, a_{i+1}, c_{i+1}\} \subseteq X$. To totally dominate $\{b_{i-1}, d_{i-1}, a_{i+1}, c_{i+1}\}$, $a_1 \in X$ if $i = 2$ and $\{a_{i-1}, c_{i-1}\} \cap X \neq \emptyset$ if $i \geqslant 3$ and $b_k \in X$ if $i = k - 1$ and $\{b_{i+1}, d_{i+1}\} \cap X \neq \emptyset$ if $i \leqslant k - 2$. But then $Y = X - \{a_{i+1}, b_{i-1}\} \cup \{a_i, b_i\}$ is a γ_t-set of G_k with $|I_Y| < |I_X|$, a contradiction. A similar argument shows that $|V_i \cap X| = 1$ is also impossible and we conclude that $\gamma_t(G_k) = 2k$. $\square$

Claim 2. *For each $k \geqslant 2$, $\gamma_t(G_k + e) = 2k - 2$ for every $e \in E(\bar{G})$.*

Proof. Let $e \in E(\bar{G})$. By symmetry, we may assume that $e = a_j b_j$ for some i, j with $1 \leqslant i < j \leqslant k$ or $e = a_j d_j$ for some i, j with $1 \leqslant i, j \leqslant k$ and $i \neq j$. Suppose $e = a_j b_j$ for some i, j with $1 \leqslant i < j \leqslant k$. Then, $(\{c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k\} - \{c_i, d_i, c_j, d_j\}) \cup \{a_i, b_j\}$ is a total dominating set of $G + e$ of cardinality $2k - 2$. Thus, by Proposition 1,

$\gamma_t(G_k + e) = 2k - 2$. Hence, assume that $e = a_i d_j$ for some i, j with $1 \leqslant i, j \leqslant k$ and $i \neq j$. We consider two possibilities.

Suppose $i < j$. Let X be the union of the set $\{c_1, d_1, \ldots, c_{i-1}, d_{i-1}\}$ if $i > 1$, the set $\{a_{j+1}, b_{j+1}, \ldots, a_k, b_k\}$ if $j < k$, the set $\{a_i, d_j\}$, and the set $\{a_{i+1}, b_{i+1}, \ldots, a_{j-1}, b_{j-1}\}$ if $i + 1 \leqslant j - 1$. Then X is a total dominating set of $G + e$ of cardinality $2k - 2$. Hence, by Proposition 1, $\gamma_t(G_k + e) = 2k - 2$.

Suppose $i > j$. Let Y be the union of the set $\{a_1, b_1, \ldots, a_{j-1}, b_{j-1}\}$ if $j > 1$, the set $\{c_{i+1}, d_{i+1}, \ldots, c_k, d_k\}$ if $i < k$, the set $\{a_i, d_j\}$, and the set $\{a_{j+1}, d_{j+1}, \ldots, a_{i-1}, d_{i-1}\}$ if $j + 1 \leqslant i - 1$. Then Y is a total dominating set of $G + e$ of cardinality $2k - 2$. Hence, by Proposition 1, $\gamma_t(G_k + e) = 2k - 2$. □

The proof of Theorem 8 now follows from Claims 1 and 2. □

If all edges between two independent sets V_i and V_{i+1} are present, then we shall say that $[V_i, V_{i+1}]$ is *full*. Next for each $k \geqslant 2$ and each $s \geqslant 2k - 1$, we construct an infinite class $\mathscr{F}_k$ of $2k$-supercritical graphs of diameter $2k - 1$. For $k \geqslant 2$, let $\mathscr{F}_k$ be the family of graphs formed from the $2k$ independent sets $V_0, V_1, \ldots, V_{2k-1}$ where $V_0 = \{a_1\}$, $V_{2k-1} = \{b_k\}$, $|V_i| \geqslant 2$ for each $i \in \{2, 3, \ldots, 2k - 2\}$, $[V_{2i}, V_{2i+1}]$ is full for each $i \in \{0, 1, \ldots, k - 1\}$, and $[V_{2i-1}, V_{2i}]$ is full minus a perfect matching for each $i \in \{1, 2, \ldots, k - 1\}$. Clearly, each graph in $\mathscr{F}_k$ is a bipartite graph with diameter $2k - 1$. Hence, each graph in $\mathscr{F}_k$ is a spanning subgraph of $K_{s,s}$ where $s = \sum_{i=1}^{k} |V_{2i-2}| \geqslant 2k - 1$. Note that when $k = 2$, each graph in the family $\mathscr{F}_2$ is obtained from $K_{s,s}$ by removing the edges of a perfect matching.

In the special case when $|V_i| = 2$ for each $i \in \{2, 3, \ldots, 2k - 2\}$ we have the graph G_k constructed earlier in this section For $i = 1, 2, \ldots, k - 1$, let b_i and d_i be distinct vertices of V_{2i-1} and let c_{i+1} and a_{i+1} be the vertices in V_{2i} that are not adjacent to b_i and d_i, respectively. For $i = 1, 2, \ldots, k$, let $W_i = V_{2i-2} \cup V_{2i-1}$. Then, a similar proof to that employed in Theorem 8 (with "V_i" replaced by "W_i") can be used to establish the following result.

Theorem 9. *For each $k \geqslant 2$, each graph in $\mathscr{F}_k$ is a $2k$-supercritical graph of diameter $2k - 1$ for some $s \geqslant 2k - 1$.*

An immediate consequence of Theorem 9 now follows.

Theorem 10. *For each $k \geqslant 2$ and each $s \geqslant 2k - 1$, there exists a $2k$-supercritical graph of diameter $2k - 1$.*

Next, we show that supercritical graphs with small diameter may have arbitrarily large total domination number. In fact, for each $k \geqslant 2$, we construct an infinite family $\mathscr{C}_k$ of $(2k + 2)$-supercritical graphs G relative to $K_{2k+1,2k+1}$ of diameter 5. For $k \geqslant 2$, form G from k copies of the cycle C_6 by identifying an edge, say ab, common to every cycle. Let $A = N(a) - \{b\}$ and $B = N(b) - \{a\}$, and label the vertices of A and B as $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ such that a_i and b_i are in the ith copy of C_6. Finally, for each $i \neq j$, add exactly one of the edges $a_i b_j$ and $a_j b_i$. Clearly, G is a bipartite spanning subgraph of $K_{2k+1,2k+1}$ and $\mathrm{diam}(G) = 5$. Since any γ_t-set of G

includes a, b, and two additional vertices from each of the k copies of C_6, it follows that $\gamma_t(G) = 2k + 2$. It is a simple exercise to check that $\gamma_t(G+e) = 2k$ for each edge $e \in E(\bar{G})$, so we omit the details of the proof to the following result.

Theorem 11. *For each* $k \geqslant 2$, $G \in \mathscr{C}_k$ *is a* $(2k+2)$*-supercritical graph relative to* $K_{2k+1,2k+1}$ *of diameter* 5.

4. *k*-Supercritical graphs for small *k*

Our aim in this section is to characterize the k-supercritical graphs for $k \in \{4, 5\}$ and to investigate properties of 6-supercritical graphs. Before proceeding further, we introduce some notation. Let G be a k-supercritical graph of diameter m. Let $u_0, u_1, \ldots, u_m$ be a diametrical path of G. For $i = 0, 1, \ldots, m$, let $V_i = \{x \mid d(u_0, x) = i\}$. Necessarily, $V_0 = \{u_0\}$, $[V_0, V_1]$ is full, and $u_i \in V_i$ for $i = 1, 2, \ldots, m$. For $i = 0, 1, \ldots, m$, let v_i denote an arbitrary vertex of V_i (possibly, $v_i = u_i$).

An immediate consequence of Lemmas 2 and 3 follows.

Theorem 12. *A graph* G *is* 4*-supercritical if and only if* $G = K_{s,s} - F$ *where* F *is a perfect matching of* G *(i.e.,* $G \in \mathscr{F}_2$*).*

Proof. The sufficiency follows from Theorem 9. Assume that G is a 4-supercritical graph. By Lemma 3, $\Delta(G) \leqslant s - 1$. Let $u \in \mathscr{L}$, and let $v \in \mathscr{R}$ be a vertex that is not adjacent to u. Then $\gamma_t(G+uv) = 2$. Moreover, by Lemma 2, $\{u, v\}$ is the unique γ_t-set of $G+uv$. Hence, $\deg_G(u) = \deg_G(v) = s-1$. It follows that each vertex of G has degree $s-1$, and so G is obtained from $K_{s,s}$ by removing the edges of a perfect matching, i.e., $G \in \mathscr{F}_2$. $\square$

Theorem 13. *There is no* 5*-supercritical graph.*

Proof. Assume that G is a 5-supercritical graph. By Lemma 3, $\Delta(G) \leqslant s - 1$. Let $uv \in E(\bar{G})$. Then $\gamma_t(G+uv) = 3$ and, by Lemma 2, every γ_t-set of $G+uv$ contains both u and v. Let S be a γ_t-set of $G+uv$ and let w be the vertex of S different from u and v. By Lemma 2, u (respectively, v) is the only vertex in that set adjacent to v (respectively, u). But then w is isolated in $\langle S \rangle$, which contradicts the fact that S is a total dominating set. Hence there is no 5-supercritical graph. $\square$

Next we consider the 6-supercritical graphs G. If s and t are non-adjacent vertices in different partite sets of G, then $\gamma_t(G+st) = 4$ and so, by Lemma 2, there exists a set T of cardinality 2 such that $T \cup \{s, t\} \succ_t G+st$. For the discussion, it is convenient to consider T to be an ordered set, the first element of which belongs to a set V_i of smallest index. That is, if $T = \{x, y\}$ where $x \in V_i$ and $y \in V_j$, then $i \leqslant j$. Furthermore, by Lemma 2, $xy \in E(G)$ (so $j = i + 1$) and neither x nor y is in $N(s) \cup N(t)$.

First we show that 6-supercritical graphs are 2-connected.

Lemma 14. *If G is a 6-supercritical graph, then G has no cutvertex.*

Proof. Let x be a cutvertex of G. We may assume that G_1 and G_2 are components of $G-x$. Let y be a neighbor of x in G_1 and z be a neighbor of x in G_2. By Lemma 6, $\delta(G)\geqslant 2$. Thus we may assume that $y'\neq x$ is in $N(y)$ and $z'\neq x$ is in $N(z)$ (so $y'\in V(G_1)$ and $z'\in V(G_2)$). Let S be a γ_t-set of $G+yz'$. By Lemma 2, S contains y and z' and neither x nor z is in S. Hence, there exists an edge ab such that either $y\succ G_1$ and $\{a,b\}\succ G_2-N[z']$ or $z'\succ G_2$ and $\{a,b\}\succ G_1-N[y]$. If $y\succ G_1$, then (since G is bipartite) y' is an endvertex, contradicting Lemma 6. If $z'\succ G_2$, then every neighbor w of z' in G_2 is either an endvertex or $w\in N(x)\cap N(z')$. Since G has no endvertices, $w\in N(x)\cap N(z')$ implying that $\{z,x,y,a,b\}\succ_t G$, a contradiction. □

Corollary 15. *If G is a 6-supercritical graph, then $|V_i|\geqslant 2$ for $1\leqslant i<\mathrm{diam}(G)$.*

Proposition 16. *If G is a 6-supercritical graph, then* $\mathrm{diam}(G)\leqslant 5$.

Proof. Suppose, to the contrary, that $\mathrm{diam}(G)\geqslant 6$. Let $u_0,u_1,\ldots,u_m$, $m\geqslant 6$, be a diametrical path of G. By Corollary 15, we know that $|V_i|\geqslant 2$ for $1\leqslant i\leqslant 5$. Consider a γ_t-set S of $G+u_1u_4$. Lemma 2 implies that both u_1 and u_4 are in S. Furthermore, the other two vertices in S must be adjacent and must dominate both V_6 and $V_1-\{u_1\}$, which is impossible. Hence, $\mathrm{diam}(G)\leqslant 5$. □

We are now in a position to characterize the 6-supercritical graphs of diameter 5.

Theorem 17. *A graph G with* $\mathrm{diam}(G)=5$ *is 6-supercritical if and only if $G\in\mathscr{F}_3$.*

Proof. The sufficiency follows from Theorem 9. To prove the necessity, assume that G is a 6-supercritical graph and $\mathrm{diam}(G)=5$. Using the notation introduced in this section the partite sets of G are $V_0\cup V_2\cup V_4$ and $V_1\cup V_3\cup V_5$. Hence, $|V_0|+|V_2|+|V_4|=s=|V_1|+|V_3|+|V_5|$.

Let S be a γ_t-set of $G+v_2v_5$. By Lemma 2, $S=\{v_2,v_5,x,y\}$ where $xy\in E(G)$ and neither x nor y is in $N(v_2)\cup N(v_5)$. Now u_0 must be dominated so without loss of generality, $x\in V_1$ and $y\in V_0\cup V_2$. Hence, $v_5\succ V_4\cup V_5$, implying that $|V_5|=1$.

Claim 3. $[V_2,V_3]$ *is full.*

Proof. Suppose $v_2v_3\in E(\bar{G})$ where $v_2\in V_2$ and $v_3\in V_3$. In any γ_t-set S of $G+v_2v_3$, both v_2 and v_3 are in S. Furthermore, the remaining two vertices of S must be adjacent and must dominate both u_0 and v_5, which is impossible. Hence, $[V_2,V_3]$ is full as claimed. □

Claim 4. $[V_1,V_2]$ *is full minus a perfect matching.*

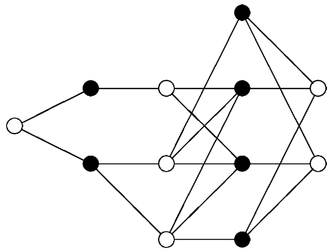

Fig. 2. A 6-supercritical graph relative to $K_{6,6}$ with diameter 4.

Proof. Since $\gamma_t(G)=6$, no vertex $v_1\in V_1$ can dominate V_2 because if it does, then $\{u_0, v_1, v_2, v_4, v_5\}$ is a total dominating set of G, a contradiction. As we have seen each vertex in V_2 is not adjacent to at least one vertex in V_1. A similar argument shows that each vertex in V_1 is not adjacent to at least one vertex in V_2. Suppose $v_1v_2\in E(\bar{G})$, and let S be a γ_t-set of $G+v_1v_2$. Then $S=\{v_1, v_2, x, y\}$ where $xy\in E(G)$. Now v_5 must be dominated, so without loss of generality, we may assume that $x\in V_4$ and $y=v_5$ (since neither x nor y is in V_3 because $V_3\in N(v_2)$). Therefore, $v_2\succ V_1-\{v_1\}$ and $v_1\succ V_2-\{v_2\}$. Hence, $[V_1, V_2]$ is full minus a perfect matching. □

Similarly, $[V_3, V_4]$ is full minus a perfect matching. Hence, $G\in\mathscr{F}_3$. This completes the proof of Theorem 17. □

To state the characterization of the 6-supercritical graphs of diameter 4, we introduce a family $\mathscr{H}$ of graphs. For integers $\ell\geqslant 2, m\geqslant 1$ and $n\geqslant 1$, let $A, B, \ldots, F$ be independent sets of vertices where $|A|=|B|=\ell$, $|C|=|D|=m$, $|E|=|F|=n$ and where each of $[A,B]$, $[C,D]$ and $[E,F]$ is full minus a perfect matching. In particular, let $\{a_1, b_1, \ldots, a_\ell b_\ell\}$ be the edges missing between A and B where $a_i\in A$ and $b_i\in B$ and let $\{c_1, d_1, \ldots, c_m, d_m\}$ be the edges missing between C and D where $c_i\in C$ and $d_i\in D$. Furthermore, $[B\cup C, E]$ is full, $[D,F]$ is full, and $[\{b_1\}, D]$ is full. We now add edges between A and C and between $B-\{b_1\}$ and D in such a way that (i) each vertex of C is adjacent to at least one vertex of A, and (ii) for $1\leqslant i\leqslant \ell$ and $1\leqslant j\leqslant m$, a_i is adjacent to c_j if and only if b_i is not adjacent to d_j. Finally, let u and v be new vertices where $N(u)=A$ and $N(v)=C\cup F$. Let $H_{\ell,m,n}$ denote the resulting graph. (Note that in $H_{\ell,m,n}$, if we let $V_i=\{x\mid d(u,x)=i\}$ for $i=0,1,\ldots,4$, then $V_0=\{u\}$, $V_1=A$, $V_2=B\cup C$, $V_3=D\cup E\cup\{v\}$, and $V_4=F$.) Fig. 2, for example, illustrates the graph $H_{2,1,2}$. Let $\mathscr{H}$ denote the family consisting of all such graphs $H_{\ell,m,n}$.

The proof of the following result is long, so we omit it. A detailed proof can be obtained from the authors.

Theorem 18. *A graph G with* $\operatorname{diam}(G)=4$ *is* 6-*supercritical if and only if* $G\in\mathscr{H}$.

Theorems 17 and 18 characterize those 6-supercritical graphs G with diameter 4 or 5. The lower bound established in Lemma 7 shows that the only remaining possibility is if $\operatorname{diam}(G)=3$. Although we have not been able to characterize the 6-supercritical graphs with minimum diameter, we know that the lower bound is sharp as can be seen with the following family of graphs. Let G be a graph with vertex set $V=A\cup B\cup C\cup D\cup\{u,v\}$ where $|A|=|B|=|C|=|D|\geqslant 2$. Let $V_1=A\cup\{v\}$. Add edges such that each of $[u,V_1]$ and $[v,C]$ is full, and each of $[A,B]$, $[B,D]$, $[A,C]$, and $[C,D]$ is full minus a perfect matching. Using the notation from this section $V_0=\{u\}$, $V_1=A\cup\{v\}$, $V_2=B\cup C$, and $V_3=D$, and it is straightforward to verify that G is 6-supercritical with diameter 3.

We conclude this section with bounds on the diameter for k-supercritical graphs where $k\in\{7,8\}$. Note that we have proven the existence of 8-supercritical graphs, but it is not clear that 7-supercritical graphs exist. Again we omit the lengthy proofs.

Proposition 19. *If G is a 7-supercritical graph, then* $\mathrm{diam}(G)\leqslant 4$.

Proposition 20. *If G is a 8-supercritical graph, then* $\mathrm{diam}(G)\leqslant 7$.

5. Open questions

In the course of this investigation, we encountered a number of problems which we have yet to settle. A partial listing of these problems follows.

1. Characterize the 6-supercritical graphs of diameter 3.
2. Does there exist a 7-supercritical graph? (If such a graph exists, then we have shown that it has diameter 3 or 4.)
3. Does there exist a k-supercritical graph with k odd?
4. Is it true that if G is a k-supercritical graph with k even, then $\mathrm{diam}(G)\leqslant k-1$? (If this is true, then the result is sharp by Theorem 10.)
5. Is it true that a graph G with $\mathrm{diam}(G)=2k-1$ is a $2k$-supercritical graph if and only if $G\in\mathscr{F}_k$? (Theorems 12 and 17 show that this is true if $k=2$ or 3.)

References

[1] P. Blitch, Domination in Graphs, Dissertation, University of South Carolina, 1983.
[2] G. Chartrand, L. Lesniak, Graphs and Digraphs, 3rd Edition, Chapman & Hall, London, 1996.
[3] E. Cockayne, Variations on the domination number of a graph, Lecture at the University of Natal, 1988.
[4] E. Cockayne, R. Dawes, S. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–219.
[5] W. Goddard, M.A. Henning, H.C. Swart, Some Nordhaus–Gaddum-type results, J. Graph Theory 16 (1992) 221–231.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[7] T.W. Haynes, M.A. Henning, Domination critical graphs with respect to relative complements, Austral. J. Combin. 18 (1998) 115–126.
[8] T.W. Haynes, C.M. Mynhardt, L.C. van der Merwe, Total domination edge critical graphs, Utilitas Math. 54 (1998) 229–240.
[9] T.W. Haynes, C.M. Mynhardt, L.C. van der Merwe, Criticality index of total domination, Congr. Numer. 131 (1998) 67–73.

[10] D.P. Sumner, Critical concepts in domination, Discrete Math. 86 (1990) 33–46.
[11] D.P. Sumner, P. Blitch, Domination critical graphs, J. Combin. Theory Ser. B 34 (1983) 65–76.
[12] D.P. Sumner, E. Wojcicka, Graphs critical with respect to the domination number, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998 (Chapter 16).
[13] E. Wojcicka, Hamiltonian properties of domination-critical graphs, J. Graph Theory 14 (1990) 205–215.